DIRECT SUM DECOMPOSITIONS AND INDECOMPOSABLE TQFT'S

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ABSTRACT. The decomposition of an arbitrary axiomatic topological quantum field theory or TQFT into indecomposable theories is given. In particular, unitary TQFT's in arbitrary dimensions are shown to decompose into a sum of theories in which the Hilbert space of the sphere is one-dimensional, and indecomposable two-dimensional theories are classified.

1. INTRODUCTION

In [DJ94], Durhuus and Jonsson define a notion of direct sum of axiomatic topological quantum field theories, or TQFT's. They show that every unitary TQFT in two dimensions can be written as a direct sum of theories in which the Hilbert space associated to the circle is one-dimensional. Such theories are easily classified and are described in terms of Euler number.

Durhuus and Jonsson leave open two questions: Higher dimensional theories and nonunitary theories. The first they explicitly address, suggesting every unitary theory in d dimensions can be decomposed into a direct sum of theories in which the Hilbert space associated to the sphere S^{d-1} is one-dimensional. In this paper we give a complete decomposition theory for TQFT's over an algebraically closed field, and describe the indecomposable theories explicitly in two dimensions. In particular we prove the conjecture suggested by Durhuus and Jonsson.

The nonunitary, indecomposable TQFT's in two dimensions which we construct are remarkably degenerate: Any cobordism of genus two or higher gets sent to the zero operator on the appropriate space. In particular, we construct many counterexamples to the conjecture that a TQFT is determined by its values on closed manifolds. The same conjecture for unitary TQFT's is still open.

This paper is divided into four sections after this introduction. Section 2 defines TQFT's and the direct sum operation of Durhuus and Jonsson. Our definition of TQFT is category theoretic, as this seems to be the most natural setting. Section 3 shows that the vector space associated with the $(d-1)$-sphere inherits the structure of a commutative Frobenius algebra from the TQFT, and most importantly that the decomposition of the TQFT into indecomposable theories is exactly the decomposition

of this Frobenius algebra into indecomposable subalgebras. It also classifies indecomposable Frobenius algebras in terms of ordinary indecomposable algebras. This, together with the parallel theory for unitary TQFT's and C^*-Frobenius algebras, which is developed along the way, gives a decomposition theory for TQFT's. Section 4 shows that two-dimensional TQFT's are determined by their Frobenius algebras, and gives a partial description of the two-dimensional indecomposable TQFT's. Section 5 ends with some remarks.

I would like to thank Michael Artin, Scott Axelrod, John Baez, Lisa Sawin, Is Singer and Washington Taylor for help and conversations. I would also like to thank John Barrett for pointing out to me the existence of non-semisimple abelian Frobenius algebras.

2. Axiomatic TQFT's and Direct Sums

The cobordism category $\mathrm{Cob}(d)$ in dimension d has as objects closed, oriented, $(d-1)$-dimensional smooth manifolds. A morphism with domain Σ_1 and codomain Σ_2, called a cobordism from Σ_1 to Σ_2, is up to diffeomorphism an oriented d-dimensional manifold with boundary $\Sigma_1^* \cup \Sigma_2$, where Σ^* is the manifold Σ with the opposite orientation. By 'up to diffeomorphism' we mean two morphisms M and M' are considered the same if there is a boundary- and orientation-preserving diffeomorphism between them. More precisely, a morphism should be a d-manifold M together with a choice of some subset of the boundary components to be the domain, and a choice of ordering of the components of the domain and codomain. We will avoid this red tape and always make the domain, codomain and order clear from context.

Composition is by gluing: If $M_1 : \Sigma_1 \to \Sigma_2$ and $M_2 : \Sigma_2 \to \Sigma_3$, then $M_2 M_1 : \Sigma_1 \to \Sigma_3$ is the manifold formed by identifying points in M_1 and M_2 on the shared boundary Σ_2. The identity $\mathbf{1}_\Sigma$ for each object Σ is the cobordism $\Sigma \times I$.

Disjoint union gives a tensor product structure on $\mathrm{Cob}(d)$. That is, we have a covariant functor from $\mathrm{Cob}(d) \times \mathrm{Cob}(d)$ to $\mathrm{Cob}(d)$ which sends $\Sigma_1 \times \Sigma_2$ to $\Sigma_1 \cup \Sigma_2$, and if $M : \Sigma_1 \to \Sigma_2$ and $M' : \Sigma_1' \to \Sigma_2'$ then $M \times M'$ goes to $M \cup M' : \Sigma_1 \cup \Sigma_1' \to \Sigma_2 \cup \Sigma_2'$. The empty $(d-1)$-manifold $\emptyset$ is the trivial object, and the empty d-manifold $\mathbf{1}_\emptyset$ is the trivial morphism. That is, they are the units for $\cup$ on objects and morphisms. Furthermore the cobordism $\mathfrak{c}_{\Sigma_1,\Sigma_2} : \Sigma_1 \cup \Sigma_2 \to \Sigma_2 \cup \Sigma_1$ given by the union of $\Sigma_1 \times I$ with $\Sigma_2 \times I$ with the boundary components ordered appropriately, satisfies

$$\mathfrak{c}_{\Sigma_1,\Sigma_2} \mathfrak{c}_{\Sigma_2,\Sigma_1} = \mathbf{1}_{\Sigma_1 \cup \Sigma_2},$$

$$\mathfrak{c}_{\Sigma_1 \cup \Sigma_2, \Gamma} = (\mathfrak{c}_{\Sigma_1, \Gamma} \otimes \mathbf{1}_{\Sigma_2})(\mathbf{1}_{\Sigma_1} \otimes \mathfrak{c}_{\Sigma_2, \Gamma})$$

and thus makes the cobordism category a symmetric monoidal or tensor category [Lan71].

$\mathrm{Vect}(\mathbb{F})$ is also a tensor category. The objects are finite-dimensional vector spaces over a field $\mathbb{F}$, and morphisms with domain V and codomain W are linear maps from V to W. Composition of morphisms is composition of linear maps, and tensor

product of objects and morphisms is tensor product of vector spaces and linear maps. Also $\mathbf{1}_V$ is the identity map on V, the trivial object is $\mathbb{F}$, and the trivial morphism is multiplication by 1 on $\mathbb{F}$.

Definition 1. [Ati89, Ati90] A TQFT is a functor $\mathcal{Z}$ of tensor categories from $\mathrm{Cob}(d)$ to $\mathrm{Vect}(\mathbb{F})$. Two TQFT's are considered *equivalent* if there is a natural isomorphism between them.

Of course, there is also a duality structure on cobordisms. If $M : \Sigma_1 \to \Sigma_2$, then we can consider M^* as a cobordism from Σ_2 to Σ_1. This kind of duality is analogous to that in the category Hil of Hilbert spaces and bounded linear functionals, where each $f : H_1 \to H_2$ has an adjoint $f^* : H_2 \to H_1$. This motivates the following strengthening:

Definition 2. A *unitary TQFT* is a tensor functor from $\mathrm{Cob}(d)$ to Hil such that $\mathcal{Z}(M^*) = \mathcal{Z}(M)^*$.

Durhuus and Jonsson [DJ94] define the notion of the direct sum of two TQFT's. The direct sum of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ is the theory $\mathcal{Z}$ which associates to each connected Σ the vector space $\mathcal{Z}_1(\Sigma) \oplus \mathcal{Z}_2(\Sigma)$, associates to each disconnected Σ the tensor product of the vector spaces associated to its components, associates to each connected M the linear map $\mathcal{Z}_1(M) \oplus \mathcal{Z}_2(M)$, interpreted in the obvious way as an operator on the appropriate vector spaces, and associates to each disconnected M the tensor product of the values of the components. The reader may check that this is again a TQFT, and that if $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are unitary, so is the direct sum.

3. Frobenius Algebras and Decomposition of TQFT's

Recall that a *Frobenius algebra* is a finite-dimensional algebra A over a field $\mathbb{F}$, together with a linear functional $\mu : A \to \mathbb{F}$ such that the bilinear pairing $(a, b) = \mu(ab)$ is nondegenerate. If the pairing is symmetric, for example if the algebra is commutative, we get what Quinn [Qui95] calls an *ambialgebra*. If A is a C^*-algebra and μ is a positive functional (i.e., $\mu(a^*a) > 0$ for all nonzero $a \in A$) then we call A a C^*-*Frobenius algebra*. The following theorem is essentially due to Dijkgraaf [Dij89]. Let S be the $(d-1)$-sphere S^{d-1}.

Proposition 1. *If $\mathcal{Z}$ is a d-dimensional TQFT, then $\mathcal{Z}$ gives $\mathcal{Z}(S)$ the structure of a commutative Frobenius algebra, and an action of this algebra on $\mathcal{Z}(\Sigma)$ for each connected $(d-1)$-manifold Σ. If $\mathcal{Z}$ is unitary, $\mathcal{Z}(S)$ is a C^*-Frobenius algebra and the action on $\mathcal{Z}(\Sigma)$ is a C^*-representation.*

Proof. Let A be $\mathcal{Z}(S)$. Multiplication is given by the map $\mathcal{Z}(M_2^1) : A \otimes A \to A$, where M_2^1 is the d-ball with two d-balls removed. The unit is the image of 1 under the map $\mathcal{Z}(M_0^1) : \mathbb{F} \to A$, where M_0^1 is the d-ball. That it is a unit is immediate, associativity follows from the fact that both sides of the associativity equation are $\mathcal{Z}$

of the d-ball with three d-balls removed. Commutativity follows from the fact that $M_2^1 = M_2^1 \mathfrak{c}_{S,S}$.

The map μ is $\mathcal{Z}(M_1^0) : A \to \mathbb{F}$, where M_1^0 is S^d with one d-ball removed. The pairing is then $\mathcal{Z}(M_2^0)$, where M_2^0 is the d-sphere with two balls removed. To see that it is nondegenerate, let M_0^2 be the connect sum of two d-balls, and notice $(\mathbf{1}_S \cup M_2^0)(M_0^2 \cup \mathbf{1}_S) = \mathbf{1}_S$, so that if $\mathcal{Z}(M_0^2)(1) = \sum_i a_i \otimes b_i \in A \otimes A$, then we have $\sum_i (x, a_i) b_i = x$, and thus the pairing is nondegenerate. See Figures 1 and 2 for a pictorial presentation of the Frobenius algebra structure and axioms.

Unit   Mu   Product   Pairing

FIGURE 1. The structure of a Frobenius algebra

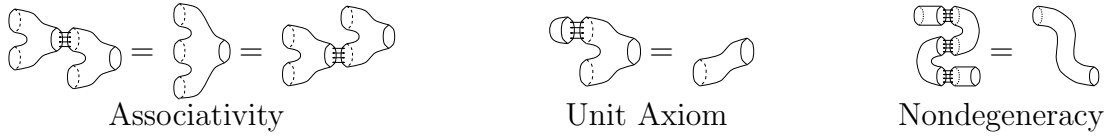Associativity   Unit Axiom   Nondegeneracy

FIGURE 2. The axioms of a Frobenius algebra

For the action of A on $\mathcal{Z}(\Sigma)$, let Σ be connected and let $M_{1,\Sigma}^{\Sigma} : S \cup \Sigma \to \Sigma$ be $\Sigma \times I$ with a d-ball removed. It is easy to check that this is an algebra action.

Now suppose that $\mathcal{Z}$ is unitary. The fact that the pairing is nondegenerate means there is a conjugate-linear isomorphism $*$ from A to itself such that $(a^*, b) = \langle a, b \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product on A. Notice that $(M_{1,\Sigma}^{\Sigma})^* = (1 \cup M_{1,\Sigma}^{\Sigma})(M_0^2 \cup 1)$, so that $\langle ax, y \rangle = \sum_i \langle a \otimes x, a_i \otimes b_i y \rangle = \sum_i \langle x, \langle a, a_i \rangle b_i y \rangle = \sum_i \langle x, (a^*, a_i) b_i y \rangle = \langle x, a^* y \rangle$. Thus A is a $*$-algebra, and the representation onto each $\mathcal{Z}(\Sigma)$ is a $*$-representation. Since this representation on $\mathcal{Z}(S)$ is faithful, A is a C^*-algebra with the operator norm in this representation, and the representations on $\mathcal{Z}(\Sigma)$ will be C^*-algebra representations. Finally $\mu(a^* a) = \langle a, a \rangle > 0$, so μ is positive. $\square$

We say that $\mathcal{Z}$ is *based on* the commutative Frobenius algebra A if $\mathcal{Z}(S)$ is isomorphic to A as a Frobenius algebra.

Theorem 1. *Suppose $\mathcal{Z}$ is based on a direct sum $A = A_1 \oplus A_2$ of Frobenius algebras. Then there exist TQFT's $\mathcal{Z}_1$ and $\mathcal{Z}_2$, based on A_1 and A_2 respectively, such that $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$. Conversely, If $\mathcal{Z}$ decomposes as a direct sum of theories, than the associated Frobenius algebra decomposes as a corresponding direct sum of Frobenius algebras. Further, the same is true for unitary TQFT's and direct sum of C^*-Frobenius algebras.*

Proof. Let p_1 and p_2 be the elements of A which correspond to the identities of A_1 and A_2 respectively, so that $p_i p_j = \delta_{i,j} p_i$ and $p_1 + p_2 = 1$. Thus if we define $\mathcal{Z}_i(\Sigma)$ for Σ connected to be the range of the action of p_i, we have that $\mathcal{Z}(\Sigma) = \mathcal{Z}_1(\Sigma) \oplus \mathcal{Z}_2(\Sigma)$. Likewise define the action of p_i on $\mathcal{Z}(\Sigma)$ for disconnected Σ to be the tensor product of its action on each connected component, and define $\mathcal{Z}_i(\Sigma)$ to be its range. Let p_i act on $\mathcal{Z}(\emptyset) = \mathbb{F}$ as 1. This defines $\mathcal{Z}$ on $(d-1)$-manifolds in a manner satisfying the assumptions of a TQFT.

Now let $M : \Sigma \to \Gamma$ be a d-cobordism. Define $M' : S^{\cup k} \cup \Sigma \to \Gamma$ to be M with a d-ball removed from each of its k components, and define $\mathcal{Z}_i(M)(x) = \mathcal{Z}(M')(p_i^{\otimes k} \otimes x)$ for $x \in \mathcal{Z}(\Sigma)$. Notice if we had removed more than one ball from some components and put p_i's in the appropriate tensor factors, we would have gotten the same operator: Removing two d-balls from the same component can be regarded as removing one d-ball and gluing M_2^1 in, so that we get the same operator as removing one ball and applying the result to $p_i \cdot p_i = p_i$. See Figure 3 for a pictorial version of this argument.
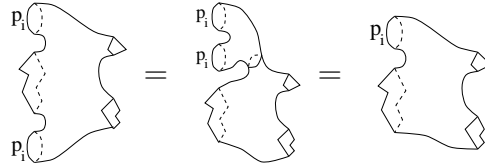

FIGURE 3. Removing two balls is the same as removing one

We claim that $\mathcal{Z}_i(M)\mathcal{Z}_i(N) = \mathcal{Z}_i(MN)$. Since this implies that $\mathcal{Z}_i(M)\mathcal{Z}(M_{1,\Sigma}^{\Sigma}) = \mathcal{Z}_i(M)\mathcal{Z}_i(\mathbf{1}_\Sigma) = \mathcal{Z}_i(M) = \mathcal{Z}(M_{1,\Gamma}^{\Gamma})\mathcal{Z}_i(M)$, we have $\mathcal{Z}_i(M) : \mathcal{Z}_i(\Sigma) \to \mathcal{Z}_i(\Gamma)$ and $\mathcal{Z}_i$ is a functor. Since $\mathcal{Z}_i(M \cup N) = \mathcal{Z}_i(M) \otimes \mathcal{Z}_i(N)$, it is a tensor functor and hence a TQFT.

To see this claim, notice that $\mathcal{Z}_i(M)\mathcal{Z}_i(N)(x)$ is $\mathcal{Z}(K)(p_i^{\otimes n} \otimes x)$, where K is MN with some positive number of d-balls removed from each component. We have already seen this is the same as $\mathcal{Z}_i(MN)$ (see Figure 4).
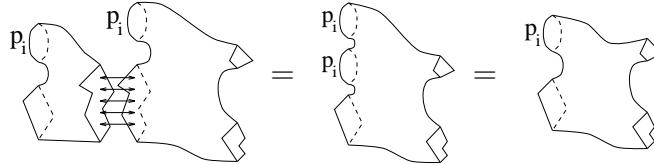

FIGURE 4. The action of p_i intertwines composition

Thus $\mathcal{Z}_i$ is a TQFT.

Is $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$? For Σ connected we have $\mathcal{Z}(\Sigma) = \mathcal{Z}_1(\Sigma) \oplus \mathcal{Z}_2(\Sigma)$, and clearly if Σ is not connected, then $\mathcal{Z}(\Sigma)$ is the tensor product of $\mathcal{Z}$ of the connected components. If M is connected, then $\mathcal{Z}_i(M)(x) = \mathcal{Z}(M')(p_i \otimes x)$, so $\mathcal{Z}_1(M)(x) + \mathcal{Z}_2(M)(x) =$

$\mathcal{Z}(M')((p_1 + p_2) \otimes x) = \mathcal{Z}(M)(x)$. Likewise, if M is disconnected then $\mathcal{Z}(M)$ is the tensor product of the values of $\mathcal{Z}$ on the connected components. Thus $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$.

The converse is easy. If $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$, then $\mu = \mathcal{Z}(M_1^0) = \mathcal{Z}_1(M_1^0) \oplus \mathcal{Z}_2(M_1^0) = \mu_1 \oplus \mu_2$. Also $\mathbf{m} = \mathcal{Z}(M_2^1) = \mathcal{Z}_1(M_2^1) \oplus \mathcal{Z}_2(M_2^1) = \mathbf{m}_1 \oplus \mathbf{m}_2$, where $\mathbf{m}$, $\mathbf{m}_1$, and $\mathbf{m}_2$ represent the products in A, A_1 and A_2 respectively. Thus $A = A_1 \oplus A_2$.

If $\mathcal{Z}$ is unitary and its C^*-Frobenius algebra A decomposes as C^*-Frobenius algebra into a direct sum $A_1 \oplus A_2$, then the above argument shows that $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$ as TQFT's: We just need to show that $\mathcal{Z}_i$ is unitary. But p_i is a self-adjoint projection, since the direct sum was as a C^*-algebra direct sum, so

$$
\begin{aligned}
\langle y, \mathcal{Z}_i(M)x \rangle &= \langle y, \mathcal{Z}(M)(p_i x) \rangle \\
&= \langle \mathcal{Z}(M)^* y, p_i x \rangle \\
&= \langle p_i \mathcal{Z}(M^*)y, x \rangle \\
&= \langle \mathcal{Z}(M^*)(p_i y), x \rangle \\
&= \langle \mathcal{Z}_i(M^*)y, x \rangle.
\end{aligned}
$$

Thus $\mathcal{Z}_i$ is unitary. Conversely, if $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$ is a direct sum of unitary theories, then the subspaces A_1 and A_2 of A are orthogonal, so the C^*-norm is the direct sum norm and the involution is the direct sum involution. $\square$

Thus the direct sum decomposition of a TQFT corresponds exactly to the direct sum decomposition of its associated commutative Frobenius algebra. Indecomposable commutative C^*-Frobenius algebras are easy to classify: For $\lambda \in \mathbb{R}^+$, define the commutative C^*-Frobenius algebra $\mathbf{C}_\lambda$ to be the C^*-algebra $\mathbb{C}$, with $\mu(x) = \lambda^{-1}x$. It is clearly a simple C^*-algebra, and μ is positive. Since the only indecomposable commutative C^*-algebra is one-dimensional, it is clear this exhausts all the possibilities.

Corollary 1. Every unitary TQFT is a direct sum of unitary TQFT's, each based on the Frobenius algebra $\mathbf{C}_\lambda$ for some λ. In particular, every unitary TQFT is the direct sum of theories with one-dimensional $\mathcal{Z}(S^{d-1})$.

The story is a bit more complicated for arbitrary Frobenius algebras, because there are so many commutative algebras. Nevertheless, we can get a fairly complete description modulo this issue. Assume $\mathbb{F}$ is algebraically closed. For each $\lambda \in \mathbb{F}$ nonzero, let $\mathbf{S}_\lambda$ be the algebra $\mathbb{F}$ with $\mu(x) = \lambda^{-1}x$. Also, let A be a commutative algebra spanned by the identity and at least one nilpotent, and suppose the *socle*, the space of all $x \in A$ such that $ax = 0$ for all nilpotent $a \in A$, is one-dimensional. Let μ be any linear functional on A which is nonzero on the socle. Let $\mathbf{N}_{A,\mu}$ be this algebra together with this functional.

Proposition 2. $\mathbf{S}_\lambda$ *and* $\mathbf{N}_{A,\mu}$ *are indecomposable Frobenius algebras. Further, every commutative indecomposable Frobenius algebra is isomorphic to one of these, and these are nonisomorphic up to algebra isomorphism.*

Proof. Obviously $\mathbf{S}_\lambda$ is an indecomposable Frobenius algebra. Now consider $\mathbf{N}_{A,\mu}$. Notice for any finite-dimensional algebra there is a bound on the number of nilpotent elements which can be multiplied to get a nonzero product. Thus for any $x \in A$, there must be a $y \in A$ such that xy is a nonzero element of the socle, for otherwise we would be able to write an arbitrarily long product of nilpotents times x which is nonzero. For this y we have $\mu(xy)$ is nonzero, and thus the bilinear form is nodegenerate. So $\mathbf{N}_{A,\mu}$ is a Frobenius algebra. It is clearly indecomposable, because A is indecomposable.

To see that every indecomposable Frobenius algebra is one of the above, first note that if a Frobenius algebra decomposes into a direct sum as an algebra, it decomposes in the same way as a Frobenius algebra, because the summands are orthogonal subspaces with respect to the bilinear pairing, and thus the bilinear pairing is nondegenerate when restricted to each. So every indecomposable Frobenius algebra is also indecomposable as an algebra. Thus it is spanned by the identity and nilpotents. If it has no nilpotents, then it is one-dimensional, and it is clearly isomorphic to exactly one $\mathbf{S}_\lambda$. If it has nilpotents, then arguing as above it has a nonempty socle. Every element of the socle is orthogonal to all nilpotent elements, so the socle is dual to the space spanned by the identity, and thus is one-dimensional. So the algebra is isomorphic to exactly one of the algebras used to construct the $\mathbf{N}$'s. Clearly μ must be nonzero on the socle, or otherwise the socle would be orthogonal to all of A. Thus the algebra isomorphism extends to a Frobenius algebra isomorphism to exactly one $\mathbf{N}_{A,\mu}$. $\quad\square$

We call a TQFT *simple* if it is based on $\mathbf{S}_\lambda$, i.e., if $\mathcal{Z}(S)$ is one-dimensional. We call a TQFT based on $\mathbf{N}_{A,\mu}$ *nilpotent*.

Corollary 2. Every TQFT is a direct sum of simple and nilpotent theories.

4. TQFT's in Two Dimensions

In two dimensions, the Frobenius algebra completely determines the TQFT. The following theorem is due to Dijkgraaf [Dij89].

Theorem 2. *There is exactly one two-dimensional TQFT based on a given commutative Frobenius algebra. There is exactly one two-dimensional unitary TQFT based on a given commutative C^*-Frobenius algebra.*

Proof. First let us construct the TQFT from a given commutative Frobenius algebra A. Certainly the vector space associated to n copies of S^1 is $A^{\otimes n}$ and to the empty one-manifold is $\mathbb{F}$. We must associate operators to two-manifolds. Clearly $\mathcal{Z}(M_2^1)$ is the product, $\mathcal{Z}(M_1^0)$ is μ, $\mathcal{Z}(M_0^1)$ is the identity, and $\mathcal{Z}(M_1^2)$ is the dual map to the product under the pairing. If M is any two-dimensional cobordism, use a Morse function to write it as a product of cobordisms, each of which is a union of some number of copies of $\mathbf{1}_S$ with one of M_1^0, M_0^1, M_1^2, M_2^1. The value of $\mathcal{Z}$ on such a cobordism is determined by the requirement of tensor functoriality and the values of

these four cobordisms. To check it does not depend on the Morse function, recall by Cerf theory [Cer70] that any change of Morse function has the effect of a sequence of the following moves, illustrated in Figure 5:

(i) $M_n^m \mathbf{1}_{S^{\cup n}} = M_n^m = \mathbf{1}_{S^{\cup m}} M_n^m$

(ii) $(M_1^0 \cup \mathbf{1}_S)(M_1^2) = (\mathbf{1}_S \cup M_1^0)(M_1^2) = \mathbf{1}_S = M_2^1(M_0^1 \cup \mathbf{1}_S) = M_2^1(\mathbf{1}_S \cup M_0^1)$

(iii) $(M_2^1 \cup \mathbf{1}_S)(\mathbf{1}_S \cup M_1^2) = M_1^2 M_2^1 = (\mathbf{1}_S \cup M_2^1)(M_1^2 \cup \mathbf{1}_S)$.
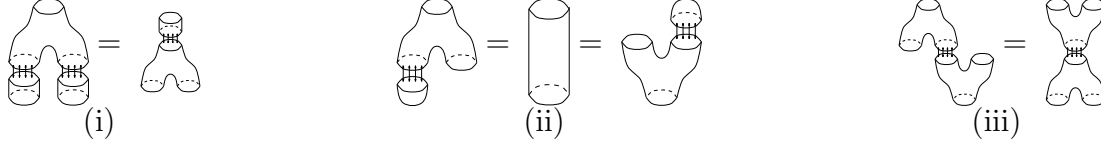

FIGURE 5. Moves of Cerf theory

Our construction of $\mathcal{Z}(M)$ is clearly invariant under (i). Move (ii) is just the statement that $1 \in A$ is the identity, and that μ is dual to the identity. For (iii) notice $\mathcal{Z}(M_1^2)(x) = \sum_i xa_i \otimes b_i$, where a_i is a basis of A and b_i is its dual basis. Then the left-hand side of (iii) on $x \otimes y$ is $\sum xya_i \otimes b_i = \mathcal{Z}(M_1^2)\mathcal{Z}(M_1^2)(x \otimes y)$, the right-hand side. Likewise $\mathcal{Z}(M_1^2)(x) = \sum a_i \otimes xb_i$, showing the other equality. Thus $\mathcal{Z}(M)$ is well-defined.

A Morse function on M_1 and M_2 gives a Morse function on $M_1 M_2$, so $\mathcal{Z}(M_1 M_2) = \mathcal{Z}(M_1)\mathcal{Z}(M_2)$. Clearly $\mathcal{Z}(\mathbf{1}_{S^{\cup n}}) = \mathbf{1}_{A^{\otimes n}}$, so $\mathcal{Z}$ is a functor. It is easy to check that it takes union of one-manifolds and cobordisms to tensor product of vector-spaces and operators respectively, the empty one- and two-manifold get sent to $\mathbb{F}$ and 1 respectively, and that permutation of components corresponds to permutation of tensor factors. Thus $\mathcal{Z}$ is a TQFT.

If $\mathcal{Z}'$ is another TQFT based on the same Frobenius algebra A, the identification of the Frobenius algebras gives a linear isomorphism between the vector spaces of $\mathcal{Z}$ and $\mathcal{Z}'$. What's more, this map intertwines $\mathcal{Z}(M_n^m)$ with $\mathcal{Z}'(M_n^m)$ with $m, n = 0, 1, 2$ as above, since their value is determined by the Frobenius algebra. The fact that $\mathcal{Z}$ and $\mathcal{Z}'$ are tensor functors then ensures that the isomorphism intertwines $\mathcal{Z}(M)$ with $\mathcal{Z}'(M)$. Since it obviously intertwines the tensor product structure, this is a natural isomorphism.

If A is a C^*-Frobenius algebra, then $\langle a, b \rangle = \mu(a^* b)$ defines a positive-definite inner product on A, and hence on every vector space associated with the theory. One need only check that $\mathcal{Z}(M_1^0) = \mathcal{Z}(M_0^1)^*$ and $\mathcal{Z}(M_1^2) = \mathcal{Z}(M_2^1)^*$. $\square$

Thus there is a two-dimensional indecomposable TQFT associated to each $\mathbf{S}_\lambda$ and each $\mathbf{N}_{A,\mu}$, and an indecomposable unitary TQFT to each $\mathbf{C}_\lambda$, and every two-dimensional TQFT is a direct sum of these.

It is worth noting what these theories look like. Following Durhuus and Jonsson [DJ94], we check that in the theory $\mathcal{Z}_\lambda$ associated to $\mathbf{S}_\lambda$, every vector space can be

associated to $\mathbb{F}$ in such a way that $\mathcal{Z}_\lambda(M) = \lambda^{-\chi(M)/2}$, where $\chi(M)$ is the Euler number.

For the nilpotent case, we will of course not be able to give a simple description of the whole TQFT $\mathcal{Z}_{A,\mu}$, but we can describe a surprisingly large amount.

We can write a chain of ideals $A = N_n \supset N_{n-1} \supset \cdots \supset N_1$, where N_1 is the socle, and each N_k is the preimage in A of the socle of A/N_{k-1}. Choose a basis for a_i of A which restricts to a basis of each N_k, and let b_i be its dual basis. We claim $a_i b_i = s$, where s is the element of the socle with $\mu(s) = 1$. To see this, notice if $a_i \in N_k$ and y is nilpotent, then $ya_i \in N_{k-1}$, and hence can be written as a linear combination of a_j for $j \neq i$. Thus $ya_i b_i = 0$, and $a_i b_i$ is in the socle. Since $\mu(a_i b_i) = 1$, we have $a_i b_i = s$.

Now let $M = M_2^1 M_1^2$, a twice-punctured torus. Then

$$\mathcal{Z}(M)(x) = \sum_i x a_i b_i = \dim(A) x s = \dim(A) f(x) s$$

where $f(x)$ is the functional on A which is 1 on the identity and zero on the nilpotents (the unique homomorphism to $\mathbb{F}$). From this one easily concludes that the torus with n incoming punctures and m outgoing punctures is sent to the operator

$$x_i \otimes \cdots \otimes x_n \mapsto f(x_1) \cdots f(x_n) \dim(A) s^{\otimes m}$$

and any manifold of genus more than one gets sent to the operator 0 on the appropriate space.

5. REMARKS

- It would be nice to find an action or state sum definition of the two-dimensional nilpotent TQFT's $\mathcal{Z}_{A,\mu}$. There is no obvious impediment to this, but the surprising behavior of this TQFT would appear to make it difficult.
- It has been asked (e.g. by [DJ94]) whether two TQFT's which agree on all closed d-manifolds are naturally isomorphic. The answer is no, even in two dimensions, if we do not restrict to unitary theories. For any $\mathcal{Z}_{A,\mu}$ the sphere gets sent to $\mu(1)$, the torus gets sent to $\dim(A)$, and all others get sent to zero. Clearly many different A and μ give the same values for these, and since they correspond to nonisomorphic Frobenius algebras, they correspond to inequivalent TQFT's. The values on closed d-manifolds does determine the TQFT for 2-dimensional unitary and semisimple theories.
- It is clear that the proofs of Proposition 1 and Theorem 2 really only involve the category of cobordisms. Thus it would be natural to express them as corollaries to purely topological theorems about this category. Specifically, we could define the notion of a Frobenius object in a tensor category, and a Frobenius action of one object on another. Then Proposition 1 follows from the statement that S is a Frobenius object in $\mathrm{Cob}(d)$ with a Frobenius action

on each connected $\sum$, and Theorem 2 from the statement that Cob(2) is the free tensor category generated by one Frobenius object.

- If $\mathcal{Z}$ is a d dimensional TQFT, and X is an r-manifold for $r < d$, then $\mathcal{Z}$ and X together naturally give a $(d-r)$-dimensional TQFT, which assigns to each $(d-r-1)$-manifold Σ the vector space $\mathcal{Z}(\Sigma \times X)$, and to each $(d-r)$-cobordism M the operator $\mathcal{Z}(M \times X)$. In particular for each $(d-2)$-manifold we get a two-dimensional TQFT, which we can classify as in the previous section. This classification should give important information about the TQFT in a simple format. For example, if we take the Chern-Simons TQFT and $X = S^1$, we get a sum of simple TQFT's with $\lambda_i = S_{0,i}^{-1}$.

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